A list of all integrable 2D homogeneous polynomial potentials with a polynomial integral of order at most 4 in the momenta

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Abstract. We searched integrable 2D homogeneous polynomial potential with a polynomial first integral by using the so-called direct method of searching for first integrals. We proved that there exist no polynomial first integrals which are genuinely cubic or quartic in the momenta if the degree of homogeneous polynomial potentials is greater than 4.

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1. Introduction

A Hamiltonian system with n degrees of freedom is integrable if the system admits n independent first integrals in involution (Liouville integrability). It is a fundamental and important problem to investigate the integrability of Hamiltonian systems. Since the Hamiltonian itself is a first integral, the case of one degree of freedom is trivial. The simplest but non-trivial problem arises in the case of two degrees of freedom. For this case, the existence of an additional first integral guarantees the integrability. Let us consider a Hamiltonian system with two degrees of freedom,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{1.1}$$

At present, there is no ultimate algorithm (necessary and sufficient conditions for integrability) to determine whether the system (1.1) is integrable or not for a given potential $V(x, y)$.

The solution of the system is said to possess the Painlevé property if it has no movable singular point other than poles. The Painlevé property of the solution has been believed to be closely related to the integrability of the system, which is known as the Painlevé conjecture (see, for example, [12, 13, 20]). Although any rigorous relation between the Painlevé property and the integrability has not been established, some new

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integrable systems were detected [5, 19] by postulating that the solution possesses the Painlevé property (the Painlevé test).

For some Hamiltonian systems of the form (1.1), it is possible to prove the non-integrability, i.e. the non-existence of an additional first integral. In the early 1980s, Ziglin [25, 26] presented a non-integrability theorem and proved the non-integrability of some well-known Hamiltonian systems. Yoshida [23] gave a criterion for non-integrability of Hamiltonian systems (1.1) with a homogeneous potential by using Ziglin's theorem [25]. Recently, the differential Galois theory has become an important tool in attacking the problem of integrability (see, for example, [2, 3]). Morales-Ruiz and Ramis [18] obtained a stronger necessary condition for integrability from their own theorem [16, 17] based on the differential Galois theory (see also [15]). This necessary condition also justified the so-called weak-Painlevé property [19, 20] as a necessary condition for integrability for the first time [24].

On the other hand, we have difficulty in proving the integrability. Even though a given system passes the Painlevé test or satisfies the necessary condition for integrability, it remains unknown whether or not the system is actually integrable. In order to prove the integrability of the system, we have to present a first integral which is independent of the Hamiltonian. However, there is no general method to obtain an explicit expression of the desired first integral. Let Φ be a first integral of the system (1.1). Then the Poisson bracket of Φ and H vanishes, which gives the following partial differential equation (PDE).

$$\frac{\partial\Phi}{\partial x}\frac{\partial H}{\partial p_x}-\frac{\partial\Phi}{\partial p_x}\frac{\partial H}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial H}{\partial p_y}-\frac{\partial\Phi}{\partial p_y}\frac{\partial H}{\partial y}=p_x\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi}{\partial p_x}\frac{\partial V}{\partial x}+p_y\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial p_y}\frac{\partial V}{\partial y}=0. \tag{1.2}$$

In the present paper, we search polynomial solutions of the PDE (1.2), i.e. we assume that the first integral is a polynomial in (x, y, p_x, p_y) and that the potential is a polynomial in (x, y). The advantage of considering polynomials is that the PDE (1.2) becomes an identity for (x, y, p_x, p_y) and that the problem reduces to completely algebraic one. In addition, we assume that the potential $V(x, y)$ is a homogeneous polynomial, motivated by the following three points: (i) Any polynomial potential $V(x, y)$ can be written in the form of the sum of homogeneous parts as

$$V(x,y)=V_{\mathrm{min}}(x,y)+\cdots+V_{\mathrm{max}}(x,y), \tag{1.3}$$

where V_{min} and V_{max} are the lowest degree part and the highest degree part, respectively; and it can be shown [10] that if the system (1.1) with the potential (1.3) admits a polynomial first integral, then the systems only with the lowest degree part and the highest degree part, given by

$$H=\frac{1}{2}(p_x^2+p_y^2)+V_{\mathrm{min}}(x,y),\quad H=\frac{1}{2}(p_x^2+p_y^2)+V_{\mathrm{max}}(x,y) \tag{1.4}$$

both admit polynomial first integrals. Namely, in order for the system with a non-homogeneous potential to be integrable, each of the systems only with the highest degree part of the potential and the lowest degree part of the potential must be integrable.

(ii) As we will see in section 3, the homogeneity of potentials assumes the weighted-homogeneity of first integrals, which simplifies the form of first integrals and reduces the complexities of computations. Indeed, the computations for non-homogeneous potentials are too complicated to deal with. (iii) The homogeneity of potentials plays an essential role to obtain the non-integrability criterions in [18, 23, 24]. For these reasons, we treat Hamiltonian systems (1.1) with a homogeneous polynomial potential of degree k,

$$V(x,y) = V_k = \sum_{j=0}^{k} \alpha_j x^{k-j} y^j = \alpha_0 x^k + \alpha_1 x^{k-1} y + \cdots + \alpha_k y^k. \tag{1.5}$$

Let us here mention the rotational degrees of freedom. The integrability is preserved under rotations of coordinates, i.e. a potential obtained from an integrable potential by a rotation of coordinates is again integrable. We should identify such two potentials. In the present paper, we assume that $\alpha_1 = 0$ from the beginning, which partially removes the rotational degrees of freedom.

Hietarinta [8] performed the direct search for integrable systems of the form (1.1), where the potential is a homogeneous polynomial of degree 5 or less and the additional first integral is a polynomial of order 4 or less in the momenta. The purpose of the present paper is to extend the degree of homogeneous polynomial potentials to an arbitrary positive integer in order to make a complete list of integrable systems in the range studied.

We have to admit that our search does not cover the whole of integrable systems. In fact, there are some integrable systems with a rational or transcendental first integral [9, 10]. More generally, as seen in the approaches based on Ziglin's analysis or the differential Galois theory, integrability requires meromorphic first integrals. However, it would be almost impossible to single out all integrable cases without any restrictions on the class of the first integral and of the potential. It may be said that restrictive assumptions are indispensable for carrying out a thorough search for integrable systems. In this sense, our setting, in which the whole computations are tractable, is the fundamental one for a thorough search.

This paper is organized as follows. The new result (theorem 2) will be shown after a brief summary of the known facts in section 2. In section 3, the statement of theorem 2 is amplified, followed by details of the computations in section 4. Finally, in section 5, we present a list of all integrable 2D homogeneous polynomial potentials with a polynomial first integral of order at most 4 in the momenta.

2. The known facts and the new result

The classification of first integrals up to quadratic in the momenta is well-known [10]. The existence of a first integral linear in the momenta is related with symmetry, the invariance of the system under a transformation: the conservation laws of momentum and angular momentum correspond to the invariances by rotations and translations,

respectively. These are particular cases of Noether's theorem [1, 11]. First integrals quadratic in the momenta are exhausted by the Bertrand-Darboux theorem:

Theorem 1 ([14]) *The following three conditions are equivalent.*
(i) A Hamiltonian system of the form (1.1) *possesses an additional first integral quadratic in the momenta.*
(ii) The potential function satisfies a linear partial differential equation called Darboux equation.
(iii) The system is separable in elliptic, polar, parabolic, or Cartesian coordinates.

If we apply the Bertrand-Darboux theorem to the potential (1.5), then we obtain the following homogeneous polynomial potentials with an additional polynomial first integral quadratic (or linear) in the momenta.

- separable in polar coordinates

$$V_k = r^k = (x^2+y^2)^{k/2} = \sum_{m=0}^{k/2} \binom{k/2}{m} x^{k-2m} y^{2m}, \quad k = \text{even} \tag{2.1a}$$

$$\Phi = yp_x - xp_y \tag{2.1b}$$

- separable in parabolic coordinates [19]

$$V_k = \frac{1}{r}\left[\left(\frac{r+x}{2}\right)^{k+1} + (-1)^k \left(\frac{r-x}{2}\right)^{k+1}\right] = \sum_{m=0}^{[k/2]} 2^{-2m} \binom{k-m}{m} x^{k-2m} y^{2m} \tag{2.2a}$$

$$\Phi = p_y(yp_x - xp_y) + \frac{1}{2} y^2 V_{k-1} \tag{2.2b}$$

- separable in Cartesian coordinates

$$V_k = Ax^k + By^k \tag{2.3a}$$

$$\Phi = p_x^2 + 2Ax^k \quad (\text{including } \Phi = p_x \text{ for } V_k = y^k) \tag{2.3b}$$

Note that the potential separable in elliptic coordinates drops out because it can not be a homogeneous polynomial. It is also possible to obtain the above list by using the direct search (see [4, 10]).

There seems to be no special property of the system connected to the existence of first integrals of higher orders in the momenta. We can make polynomial first integrals of *apparently* higher orders in the momenta from the above polynomial first integrals (2.1b), (2.2b), and (2.3b). For example,

$$\Phi = (yp_x - xp_y)^4, \quad \Phi = \left\{p_y(yp_x - xp_y) + \frac{1}{2} y^2 V_{k-1}\right\}^2, \quad \Phi = (p_x^2 + 2Ax^k)^2 \tag{2.4}$$

are polynomial first integrals which are *apparently* quartic in the momenta for the potentials (2.1a), (2.2a), (2.3a), respectively. On the other hand, Hall [7] and Grammaticos *et al* [5] found independently the potential of degree 3,

$$V_3 = x^3 + \frac{3}{16} xy^2 \tag{2.5a}$$

with an additional first integral

$$\varPhi = p_y^4 - \frac{1}{4}y^3 p_x p_y + \frac{3}{4}xy^2 p_y^2 - \frac{3}{64}x^2 y^4 - \frac{1}{128}y^6. \tag{2.5b}$$

Ramani *et al* [19] found the potential of degree 3,

$$V_3 = x^3 + \frac{1}{2}xy^2 + \frac{\sqrt{3}\,i}{18}y^3 \tag{2.6a}$$

with an additional first integral

$$\Phi = p_x p_y^3 - \frac{\sqrt{3}\,i}{2}p_y^4 + \frac{1}{2}y^3 p_x^2 - \left(\frac{3}{2}xy^2 - \frac{\sqrt{3}\,i}{2}y^3\right)p_x p_y + \left(3x^2 y - \sqrt{3}\,ixy^2 + \frac{1}{2}y^3\right)p_y^2$$
$$+ \frac{1}{2}x^3 y^3 + \frac{\sqrt{3}\,i}{8}x^2 y^4 - \frac{1}{4}xy^5 + \frac{5\sqrt{3}\,i}{72}y^6 \tag{2.6b}$$

and the potential of degree 4,

$$V_4 = x^4 + \frac{3}{4}x^2 y^2 + \frac{1}{8}y^4 \tag{2.7a}$$

with an additional first integral

$$\varPhi = p_y^4 + \frac{1}{2}y^4 p_x^2 - 2xy^3 p_x p_y + \left(3x^2 y^2 + \frac{1}{2}y^4\right)p_y^2 + \frac{1}{4}x^4 y^4 + \frac{1}{4}x^2 y^6 + \frac{1}{16}y^8. \tag{2.7b}$$

All of the three polynomial first integrals (2.5*b*), (2.6*b*), (2.7*b*) are *genuinely* quartic in the momenta. Here, we mean by '*genuinely*' that these first integrals cannot be reduced to first integrals of lower orders in the momenta. See also [6] for the discoveries of these three integrable cases.

Now the following question arises. *Are there any other potentials that admit a polynomial first integral which is genuinely quartic (or cubic) in the momenta?* Hietarinta [8] searched integrable 2D homogeneous polynomial potentials of degree 5 or less with a polynomial first integral of order 4 or less in the momenta by means of the so-called direct method and concluded that there was no integrable case other than the known ones in the range studied. We extended the degree of homogeneous polynomial potentials to an arbitrary positive integer. Specifically, we investigated the existence of polynomial first integrals which are cubic or quartic in the momenta for Hamiltonian systems of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \alpha_0 x^k + \alpha_2 x^{k-2}y^2 + \cdots + \alpha_k y^k, \tag{2.8}$$

with the degree of the potential $k \geq 3$ (the cases for $k = 1, 2$ are obviously integrable). Note that the term of $x^{k-1}y$ in the potential vanishes for the removal of rotational degrees of freedom as mentioned in section 1. As a result, we obtained the following theorem.

Theorem 2 *If $k \geq 5$, then the Hamiltonian system* (2.8) *admits no polynomial first integrals which are genuinely cubic or quartic in the momenta.*

Therefore, the answer to the above question is "No."

3. Amplification of the statement of theorem 2

In this section, we give details of the statement of theorem 2. Without loss of generality, we can assume that a polynomial first integral $\varPhi$ has the following properties.

Property 1. A polynomial first integral $\varPhi$ is either even or odd in the momenta (p_x, p_y).
Property 2. A polynomial first integral $\varPhi$ is weighted-homogeneous, i.e. $\varPhi$ satisfies

$$\varPhi(\sigma^{2/(k-2)}x, \sigma^{2/(k-2)}y, \sigma^{k/(k-2)}p_x, \sigma^{k/(k-2)}p_y) = \sigma^M \varPhi(x, y, p_x, p_y), \tag{3.1}$$

where σ is an arbitrary constant and M is a constant called a *weight*. The property 1 is due to the time reflection symmetry of the system. The property 2 is due to the scale-invariance of the system, which arises from the homogeneity of potentials. See Appendix A for more details. It is easy to check that all the polynomial first integrals in section 2 satisfy the properties 1 and 2.

3.1. Polynomial first integrals cubic in the momenta

From the property 1, we can put a polynomial first integral which is cubic in the momenta in the form

$$\begin{aligned}\varPhi = A_0(x,y)p_x^3 + A_1(x,y)p_x^2p_y + A_2(x,y)p_xp_y^2 + A_3(x,y)p_y^3 \\ + B_0(x,y)p_x + B_1(x,y)p_y.\end{aligned} \tag{3.2}$$

If we regard the PDE (1.2) as an identity for the momenta (p_x, p_y), then we have the following three sets of PDEs.

$$\begin{cases} A_{0x} = 0 \\ A_{0y} + A_{1x} = 0 \\ A_{1y} + A_{2x} = 0 \\ A_{2y} + A_{3x} = 0 \\ A_{3y} = 0 \end{cases} \tag{3.3}$$

$$\begin{cases} B_{0x} = 3A_0V_x + A_1V_y \\ B_{0y} + B_{1x} = 2A_1V_x + 2A_2V_y \\ B_{1y} = A_2V_x + 3A_3V_y \end{cases} \tag{3.4}$$

$$B_0V_x + B_1V_y = 0 \tag{3.5}$$

where the subscripts x and y denote partial derivatives. From the property 2, the polynomials $A_i(x, y)$ are homogeneous of the same degree and so are the polynomials $B_i(x, y)$. The PDEs (3.3) have four homogeneous polynomial solutions, which classify the leading part of the first integral (3.2) into the following four cases.

Case 1. $\varPhi = a_3(yp_x - xp_y)^3 + B_0p_x + B_1p_y$
Case 2. $\varPhi = (a_2p_x + b_2p_y)(yp_x - xp_y)^2 + B_0p_x + B_1p_y$
Case 3. $\varPhi = (a_1p_x^2 + b_1p_xp_y + c_1p_y^2)(yp_x - xp_y) + B_0p_x + B_1p_y$
Case 4. $\varPhi = a_0p_x^3 + b_0p_x^2p_y + c_0p_xp_y^2 + d_0p_y^3 + B_0p_x + B_1p_y$

Let us next consider the PDEs (3.4). We obtain the PDE for $V(x,y)$,

$$\begin{aligned}&A_2V_{xxx}+(3A_3-2A_1)V_{xxy}+(3A_0-2A_2)V_{xyy}+A_1V_{yyy}\\&+2(A_{2x}-A_{1y})(V_{xx}-V_{yy})+2(3A_{0y}-A_{1x}-A_{2y}+3A_{3x})V_{xy}\\&+(3A_{0yy}-2A_{1xy}+A_{2xx})V_x+(A_{1yy}-2A_{2xy}+3A_{3xx})V_y=0\end{aligned}\tag{3.6}$$

by using

$$\begin{aligned}&\partial_y^2(3A_0V_x+A_1V_y)-\partial_x\partial_y(2A_1V_x+2A_2V_y)+\partial_x^2(A_2V_x+3A_3V_y)\\&=\partial_y^2B_{0x}-\partial_x\partial_y(B_{0y}+B_{1x})+\partial_x^2B_{1y}\\&=0.\end{aligned}\tag{3.7}$$

For the above each case, the PDE (3.6) becomes

Case 1

$$\begin{aligned}&a_3\{x^2yV_{xxx}+(2xy^2-x^3)V_{xxy}+(y^3-2x^2y)V_{xyy}-xy^2V_{yyy}\\&\qquad+8xyV_{xx}+8(y^2-x^2)V_{xy}-8xyV_{yy}+12yV_x-12xV_x\}=0,\end{aligned}\tag{3.8a}$$

Case 2

$$\begin{aligned}&a_2\{x^2V_{xxx}+4xyV_{xxy}+(3y^2-2x^2)V_{xyy}-2xyV_{yyy}\\&\qquad+8xV_{xx}+16yV_{xy}-8xV_{yy}+12V_x\}+\\&b_2\{-2xyV_{xxx}+(3x^2-2y^2)V_{xxy}+4xyV_{xyy}+y^2V_{yyy}\\&\qquad-8yV_{xx}+16xV_{xy}+8yV_{yy}+12V_y\}=0,\end{aligned}\tag{3.8b}$$

Case 3

$$\begin{aligned}&a_1(2xV_{xxy}+3yV_{xyy}-xV_{yyy}+8V_{xy})+\\&b_1(-xV_{xxx}-2yV_{xxy}+2xV_{xyy}+yV_{yyy}-4V_{xx}+4V_{yy})+\\&c_1(yV_{xxx}-3xV_{xxy}-2yV_{xyy}-8V_{xy})=0,\end{aligned}\tag{3.8c}$$

Case 4

$$a_0(3V_{xyy})+b_0(-2V_{xxy}+V_{yyy})+c_0(V_{xxx}-2V_{xyy})+d_0(3V_{xxy})=0.\tag{3.8d}$$

The potential must satisfy the two PDEs (3.6) and (3.5). They are transformed into identities for (x,y) by the substitution of the potential (1.5). Therefore we finally obtain two sets of simultaneous algebraic equations for the coefficients of the potential and of the first integral. We searched their solutions for $k\geq 3$ under the condition that the leading part of the first integral does not vanish, i.e. $a_3\neq 0$, $(a_2,b_2)\neq(0,0)$, $(a_1,b_1,c_1)\neq(0,0,0)$, or $(a_0,b_0,c_0,d_0)\neq(0,0,0,0)$. Table 1 shows the results for the four cases.

3.2. Polynomial first integrals quartic in the momenta

From the property 1, we can put a polynomial first integral which is quartic in the momenta in the form

$$\begin{aligned}\Phi=&A_0(x,y)p_x^4+A_1(x,y)p_x^3p_y+A_2(x,y)p_x^2p_y^2+A_3(x,y)p_xp_y^3+A_4(x,y)p_y^4\\&+B_0(x,y)p_x^2+B_1(x,y)p_xp_y+B_2(x,y)p_y^2+C_0(x,y).\end{aligned}\tag{3.9}$$

Table 1. Integrable potentials with a first integral cubic in the momenta.

Case	Potential	First integral
1	$V_k = r^k = (x^2+y^2)^{k/2}, \quad k = \text{even}$	$\Phi = (yp_x - xp_y)^3$
2	$V_k \equiv 0$	
3	$V_k = r^k = (x^2+y^2)^{k/2}, \quad k = \text{even}$	$\Phi = (yp_x - xp_y)H$
4	$V_k = x^k$	$\Phi = b_0 p_y(p_x^2 + 2x^k) + d_0 p_y^3$

If we regard the PDE (1.2) as an identity for the momenta (p_x, p_y), then we obtain the following three sets of PDEs.

$$\begin{cases} A_{0x} = 0 \\ A_{0y} + A_{1x} = 0 \\ A_{1y} + A_{2x} = 0 \\ A_{2y} + A_{3x} = 0 \\ A_{3y} + A_{4x} = 0 \\ A_{4y} = 0 \end{cases} \tag{3.10}$$

$$\begin{cases} B_{0x} = 4A_0V_x + A_1V_y \\ B_{0y} + B_{1x} = 3A_1V_x + 2A_2V_y \\ B_{1y} + B_{2x} = 2A_2V_x + 3A_3V_y \\ B_{2y} = A_3V_x + 4A_4V_y \end{cases} \tag{3.11}$$

$$\begin{cases} C_{0x} = 2B_0V_x + B_1V_y \\ C_{0y} = B_1V_x + 2B_2V_y \end{cases} \tag{3.12}$$

From the property 2, the polynomials $A_i(x, y)$ are homogeneous of the same degree and so are the polynomials $B_i(x, y)$ and $C_0(x, y)$. The PDEs (3.10) have five homogeneous polynomial solutions, which classify the leading part of the first integral (3.9) into the following five cases.

Case 1. $\Phi = a_4(yp_x - xp_y)^4 + B_0p_x^2 + B_1p_xp_y + B_2p_y^2 + C_0$
Case 2. $\Phi = (a_3p_x + b_3p_y)(yp_x - xp_y)^3 + B_0p_x^2 + B_1p_xp_y + B_2p_y^2 + C_0$
Case 3. $\Phi = (a_2p_x^2 + b_2p_xp_y + c_2p_y^2)(yp_x - xp_y)^2 + B_0p_x^2 + B_1p_xp_y + B_2p_y^2 + C_0$
Case 4. $\Phi = (a_1p_x^3 + b_1p_x^2p_y + c_1p_xp_y^2 + d_1p_y^3)(yp_x - xp_y) + B_0p_x^2 + B_1p_xp_y + B_2p_y^2 + C_0$
Case 5. $\Phi = a_0p_x^4 + b_0p_x^3p_y + c_0p_x^2p_y^2 + d_0p_xp_y^3 + e_0p_y^4 + B_0p_x^2 + B_1p_xp_y + B_2p_y^2 + C_0$

Let us next consider the PDEs (3.11). We obtain the PDE for $V(x, y)$,

$$\begin{aligned}
&A_3V_{xxxx} - 2(A_2 - 2A_4)V_{xxxy} + 3(A_1 - A_3)V_{xxyy} - 2(2A_0 - A_2)V_{xyyy} - A_1V_{yyyy} \\
&- (2A_{2y} - 3A_{3x})V_{xxx} + (6A_{1y} - 4A_{2x} - 3A_{3x} + 12A_{4x})V_{xxy} \\
&- (12A_{0y} - 3A_{1x} - 4A_{2y} + 6A_{3x})V_{xyy} - (3A_{1y} - 2A_{2x})V_{yyy} \\
&+ (3A_{1yy} - 4A_{2xy} + 3A_{3xx})(V_{xx} - V_{yy}) \\
&- (12A_{0yy} - 6A_{1xy} + 2A_{2xx} - 2A_{2yy} + 6A_{3xy} - 12A_{4xx})V_{xy} \\
&- (4A_{0yyy} - 3A_{1xyy} + 2A_{2xxy} - A_{3xxx})V_x
\end{aligned}$$

$$-(A_{1yyy}-2A_{2xyy}+3A_{3xxy}-4A_{4xxx})V_y=0 \tag{3.13}$$

by using

$$\begin{aligned}\partial_y^3(4A_0V_x+A_1V_y)-\partial_x\partial_y^2(3A_1V_x+2A_2V_y)+\partial_x^2\partial_y(2A_2V_x+3A_3V_y)-\partial_x^3(A_3V_x+4A_4V_y)\\ =\partial_y^3B_{0x}-\partial_x\partial_y^2(B_{0y}+B_{1x})+\partial_x^2\partial_y(B_{1y}+B_{2x})-\partial_x^3B_{2y}\\ =0.\end{aligned} \tag{3.14}$$

For the above each case, the PDE (3.13) becomes

Case 1

$$\begin{aligned}a_4\{&x^3yV_{xxxx}-(x^4-3x^2y^2)V_{xxxy}-3(x^3y-xy^3)V_{xxyy}+(y^4-3x^2y^2)V_{xyyy}-xy^3V_{yyyy}\\ &+15x^2yV_{xxx}-15(x^3-2xy^2)V_{xxy}+15(y^3-2x^2y)V_{xyy}-15xy^2V_{yyy}\\ &+60xyV_{xx}-60(x^2-y^2)V_{xy}-60xyV_{yy}+60yV_x-60xV_y\}=0,\end{aligned} \tag{3.15a}$$

Case 2

$$\begin{aligned}a_3\{&x^3V_{xxxx}+6x^2yV_{xxxy}-3(x^3-3xy^2)V_{xxyy}-2(3x^2y-2y^3)V_{xyyy}-3xy^2V_{yyyy}\\ &+15x^2V_{xxx}+60xyV_{xxy}-15(2x^2-3y^2)V_{xyy}-30xyV_{yyy}\\ &+60xV_{xx}+120yV_{xy}-60xV_{yy}+60V_x\}+\\ b_3\{&-3x^2yV_{xxxx}+2(2x^3-3xy^2)V_{xxxy}+3(3x^2y-y^3)V_{xxyy}+6xy^2V_{xyyy}+y^3V_{yyyy}\\ &-30xyV_{xxx}+15(3x^2-2y^2)V_{xxy}+60xyV_{xyy}+15y^2V_{yyy}\\ &-60yV_{xx}-120xV_{xy}+60yV_{yy}+60V_y\}=0,\end{aligned} \tag{3.15b}$$

Case 3

$$\begin{aligned}a_2\{&2x^2V_{xxxy}+6xyV_{xxyy}-2(x^2-2y^2)V_{xyyy}-2xyV_{yyyy}\\ &+20xV_{xxy}+30yV_{xyy}-10xV_{yyy}+40V_{xy}\}+\\ b_2\{&-x^2V_{xxxx}-4xyV_{xxxy}+3(x^2-y^2)V_{xxyy}+4xyV_{xyyy}+y^2V_{yyyy}\\ &-10xV_{xxx}-20yV_{xxy}+20xV_{xyy}+10yV_{yyy}-20V_{xx}+20V_{yy}\}+\\ c_2\{&2xyV_{xxxx}-2(2x^2-y^2)V_{xxxy}-6xyV_{xxyy}-2y^2V_{xyyy}\\ &+10yV_{xxx}+30xV_{xxy}-20yV_{xyy}-40V_{xy}\}=0,\end{aligned} \tag{3.15c}$$

Case 4

$$\begin{aligned}&a_1(3xV_{xxyy}+4yV_{xyyy}-xV_{yyyy}+15V_{xyy})+\\ &b_1(-2xV_{xxxy}-3yV_{xxyy}+2xV_{xyyy}+yV_{yyyy}-10V_{xxy}+5V_{yyy})+\\ &c_1(xV_{xxxx}+2yV_{xxxy}-3xV_{xxyy}-2yV_{xyyy}+5V_{xxx}-10V_{xyy})+\\ &d_1(-yV_{xxxx}+4xV_{xxxy}+3yV_{xxyy}+15V_{xxy})=0,\end{aligned} \tag{3.15d}$$

Case 5

$$\begin{aligned}&a_0(4V_{xyyy})+b_0(-3V_{xxyy}+V_{yyyy})+c_0(2V_{xxxy}-2V_{xyyy})\\ &+d_0(-V_{xxxx}+3V_{xxyy})+e_0(-4V_{xxxy})=0.\end{aligned} \tag{3.15e}$$

We also obtain the PDEs for $V(x,y)$,

$$B_1(V_{xx}-V_{yy})+2(B_2-B_0)V_{xy}+(B_{1x}-2B_{0y})V_x+(2B_{2x}-B_{1y})V_y=0 \tag{3.16}$$

Table 2. Integrable potentials with a first integral quartic in the momenta.

Case	Potential	First integral
1	$V_k = r^k = (x^2+y^2)^{k/2}, \quad k = \text{even}$	$\Phi = (yp_x - xp_y)^4$
2	$V_k \equiv 0$	
3	$V_k = r^k = (x^2+y^2)^{k/2}, \quad k = \text{even}$	$\Phi = (yp_x - xp_y)^2 H$
	$V_k = \frac{1}{r}\left[\left(\frac{r+x}{2}\right)^{k+1} + (-1)^k \left(\frac{r-x}{2}\right)^{k+1}\right]$	$\Phi = \left(p_y(yp_x - xp_y) + \frac{1}{2}y^2 V_{k-1}\right)^2$
4	$V_k = \frac{1}{r}\left[\left(\frac{r+x}{2}\right)^{k+1} + (-1)^k \left(\frac{r-x}{2}\right)^{k+1}\right]$	$\Phi = \left(p_y(yp_x - xp_y) + \frac{1}{2}y^2 V_{k-1}\right) H$
5	$V_k = Ax^k + By^k$	$\Phi = a_0(p_x^2 + 2Ax^k)^2 + e_0(p_y^2 + 2By^k)^2$

from (3.12) by using

$$\partial_y(2B_0V_x + B_1V_y) - \partial_x(B_1V_x + 2B_2V_y) = \partial_y C_{0x} - \partial_x C_{0y} = 0. \tag{3.17}$$

The potential must satisfy the two PDEs (3.13) and (3.16). They are transformed into identities for (x, y) by the substitution of the potential (1.5). Therefore, we finally obtain two sets of simultaneous algebraic equations for the coefficients of the potential and of the first integral. We searched their solutions for $k \geq 3$ under the condition that the leading part of the first integral does not vanish, i.e. $a_4 \neq 0$, $(a_3, b_3) \neq (0, 0)$, $(a_2, b_2, c_2) \neq (0, 0, 0)$, $(a_1, b_1, c_1, d_1) \neq (0, 0, 0, 0)$, or $(a_0, b_0, c_0, d_0, e_0) \neq (0, 0, 0, 0, 0)$. Table 2 shows the results obtained from the five cases, which hold for $k \geq 3$ except that there exist three exceptional potentials, (2.5a), (2.6a), (2.7a), in the case 5 for $k = 3, 4$.

All the potentials in tables 1 and 2 are separable ones given in section 2 and the corresponding first integrals are *apparently* cubic and quartic in the momenta, i.e. there are no new integrable cases. This is what the statement of theorem 2 means.

4. Details of the computations

All of the three polynomial first integrals (2.5b), (2.6b), (2.7b), which are *genuinely* quartic in the momenta, fall into the case 5. So, it is quite natural to expect that new integrable potentials, if any, would have first integrals which belong to the case 5. For this reason, we take the case 5 as an example to show the details of the computations performed in this study. We first exclude the square of the Hamiltonian by considering $\Phi - 2c_0H^2$ instead of Φ itself, i.e. we put $c_0 = 0$ from the beginning. The PDEs (3.11) for this case become

$$\begin{cases} B_{0x} = 4a_0V_x + b_0V_y \\ B_{0y} + B_{1x} = 3b_0V_x \\ B_{1y} + B_{2x} = 3d_0V_y \\ B_{2y} = d_0V_x + 4e_0V_y \end{cases} \tag{4.1}$$

We obtain the expressions of B_0, B_1, B_2 by integrating (4.1) as follows (note that $\alpha_1 = 0$).

$$B_0 = \sum_{j=0}^{k-1} \frac{4(k-j)\alpha_j a_0 + (j+1)\alpha_{j+1} b_0}{k-j} x^{k-j} y^j + r_0 y^k, \tag{4.2}$$

$$\begin{aligned} B_1 &= r_1 x^k - k r_2 x^{k-1} y \\ &\quad + \sum_{j=2}^{k} \left[\left\{ 3\alpha_j - \frac{(k-j+1)(k-j+2)}{j(j-1)} \alpha_{j-2} \right\} d_0 - \frac{4(k-j+1)}{j} \alpha_{j-1} e_0 \right] x^{k-j} y^j, \end{aligned} \tag{4.3}$$

$$B_2 = r_2 x^k + \sum_{j=1}^{k} \frac{(k-j+1)\alpha_{j-1} d_0 + 4j\alpha_j e_0}{j} x^{k-j} y^j, \tag{4.4}$$

where

$$r_0 = \frac{3(k-1)(k-2)\alpha_{k-1} b_0 + \{6\alpha_{k-3} - 3(k-1)(k-2)\alpha_{k-1}\} d_0 + 8(k-2)\alpha_{k-2} e_0}{k(k-1)(k-2)}, \tag{4.5}$$

$$r_1 = \frac{8(k-2)\alpha_2 a_0 + 6\alpha_3 b_0}{k(k-1)(k-2)}, \tag{4.6}$$

$$r_2 = \frac{\{3k(k-1)\alpha_0 - 2\alpha_2\} b_0}{k(k-1)}. \tag{4.7}$$

The PDE (3.13), or (3.15e) becomes

$$4a_0 V_{xyyy} - b_0(3V_{xxyy} - V_{yyyy}) - d_0(V_{xxxx} - 3V_{xxyy}) - 4e_0 V_{xxxy} = 0. \tag{4.8}$$

From (4.8) and (3.16) with B_0, B_1, B_2 given above, we obtain two sets of simultaneous algebraic equations for the coefficients of the potential, α_j, and of the leading part of the first integral, a_0, b_0, d_0, e_0. The simultaneous algebraic equations obtained from (4.8) consist of the following $k-3$ equations.

$$\begin{aligned} &4(j-1)j(j+1)(k-j-1)\alpha_{j+1} a_0 \\ &+ j(j-1)\{(j+1)(j+2)\alpha_{j+2} - 3(k-j)(k-j-1)\alpha_j\} b_0 \\ &+ (k-j)(k-j-1)\{3j(j-1)\alpha_j - (k-j+1)(k-j+2)\alpha_{j-2}\} d_0 \\ &- 4(j-1)(k-j-1)(k-j)(k-j+1)\alpha_{j-1} e_0 = 0 \quad (j = 2, 3, \ldots, k-2) \end{aligned} \tag{4.9}$$

which can be regarded as a recurrence relation for α_j. The simultaneous algebraic equations obtained from (3.16) consist of $2k-1$ equations of the form

$$\begin{cases} M_{11} b_0 = 0 \\ M_{21} b_0 + M_{22} a_0 = 0 \\ M_{31} b_0 + M_{32} a_0 + M_{33} d_0 = 0 \\ M_{41} b_0 + M_{42} a_0 + M_{43} d_0 + M_{44} e_0 = 0 \\ M_{51} b_0 + M_{52} a_0 + M_{53} d_0 + M_{54} e_0 = 0 \\ M_{61} b_0 + M_{62} a_0 + M_{63} d_0 + M_{64} e_0 = 0 \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{2k-1,1} b_0 + M_{2k-1,2} a_0 + M_{2k-1,3} d_0 + M_{2k-1,4} e_0 = 0 \end{cases} \tag{4.10}$$

where the first four equations are given by the following.

$$M_{11} = \frac{\{3k(2k-1)\alpha_0 - 2\alpha_2\}\{k(k-1)\alpha_0 - 2\alpha_2\}}{k(k-1)},$$

$$M_{21}=-\frac{6(7k-6)\alpha_0\alpha_3}{k-2}+\frac{12(7k-6)\alpha_2\alpha_3}{k(k-1)(k-2)},$$

$$M_{22}=-\frac{16(3k-2)\{k(k-1)\alpha_0-2\alpha_2\}\alpha_2}{k(k-1)},$$

$$M_{31}=6(k-1)(2k-3)\alpha_0\alpha_2-\frac{2(17k^2-28k+6)\alpha_2^2}{k(k-1)}+\frac{18(7k-6)\alpha_3^2}{k(k-1)(k-2)}$$
$$-\frac{12(7k^2-22k+18)\alpha_0\alpha_4}{(k-2)(k-3)}+\frac{48(2k^2-4k+3)\alpha_2\alpha_4}{k(k-1)(k-2)(k-3)},$$

$$M_{32}=-\frac{12(2k-3)(3k-4)\alpha_0\alpha_3}{k-2}+\frac{24(2k-3)(5k-4)\alpha_2\alpha_3}{k(k-1)(k-2)},$$

$$M_{33}=4k(2k-3)\alpha_0\alpha_2,$$

$$M_{41}=6(k-2)(2k-3)\alpha_0\alpha_3-\frac{4(29k^2-44k+6)\alpha_2\alpha_3}{k(k-1)}+\frac{48(7k-12)\alpha_3\alpha_4}{k(k-2)(k-3)}$$
$$-\frac{20(7k^2-31k+36)\alpha_0\alpha_5}{(k-3)(k-4)}+\frac{40(5k^2-11k+12)\alpha_2\alpha_5}{k(k-1)(k-3)(k-4)},$$

$$M_{42}=-\frac{16(k-2)(3k-4)\alpha_2^2}{k-1}+\frac{144\alpha_3^2}{k-2}-\frac{96(k-2)^2\alpha_0\alpha_4}{k-3}+\frac{384(k-2)\alpha_2\alpha_4}{k(k-3)},$$

$$M_{43}=12k(k-2)\alpha_0\alpha_3,\quad M_{44}=32(k-2)\alpha_2^2$$

Now, what we have to do is to find solutions of (4.9) and (4.10) under the condition $(b_0,a_0,d_0,e_0)\neq(0,0,0,0)$. Applying this condition to the first four equations of (4.10), we see that the product $M_{11}M_{22}M_{33}M_{44}$ must vanish. We therefore have the following possibilities for the relation between α_0 and α_2.

$$\text{(i) } \alpha_0=0\quad \text{(ii) } \alpha_2=0\quad \text{(iii) } \alpha_2=\frac{k(k-1)}{2}\alpha_0\quad \text{(iv) } \alpha_2=\frac{3k(2k-1)}{2}\alpha_0 \tag{4.11}$$

Once the relation between α_0 and α_2 is given, the computations to follow are quite straightforward. Let us move on to the details of each case.

(i) When $\alpha_0=0$, the first and the second equations of (4.10) become

$$\begin{cases}\dfrac{4\alpha_2^2}{k(k-1)}b_0=0\\ \dfrac{12(7k-6)\alpha_2\alpha_3}{k(k-1)(k-2)}b_0+\dfrac{32(3k-2)\alpha_2^2}{k(k-1)}a_0=0\end{cases} \tag{4.12}$$

If we assume that $\alpha_2\neq0$ then we see that $b_0=a_0=0$, and then we can show from (4.9) with $j=2,3$ that $d_0=e_0=0$. This contradicts the condition $(b_0,a_0,d_0,e_0)\neq(0,0,0,0)$. Therefore, α_2 must vanish. Then the third equation of (4.10) becomes

$$\frac{18(7k-6)\alpha_3^2}{k(k-1)(k-2)}b_0=0. \tag{4.13}$$

Let us assume that $\alpha_3\neq0$. Then, $b_0=0$ and we can show from (4.9) with $j=2,3,4$ that $a_0=d_0=e_0=0$. This contradicts the condition $(b_0,a_0,d_0,e_0)\neq(0,0,0,0)$. Therefore, α_3 must vanish. Let us now suppose that we have proved that $\alpha_0=\alpha_1=\cdots=\alpha_j=0$ $(j\geq3)$. Then it follows from (4.9) that α_{j+1} must vanish, up to $j=k-4$, under the

condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$. Now, we have $\alpha_0 = \alpha_1 = \cdots = \alpha_{k-3} = 0$. If we assume that $\alpha_{k-2} \neq 0$ then we have $b_0 = a_0 = d_0 = 0$ from (4.9) with $j = k-4, k-3, k-2$. Then the $(2k-4)$-th equation of (4.10) becomes

$$\frac{16(k-2)(3k-4)\alpha_{k-2}^2}{k-1} e_0 = 0 \tag{4.14}$$

which shows that $e_0 = 0$. This contradicts the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$. Therefore, α_{k-2} must vanish. Let us assume that $\alpha_{k-1} \neq 0$. Then we have $b_0 = a_0 = 0$ from (4.9) with $j = k-3, k-2$. The $(2k-3)$-rd and the $(2k-2)$-nd equations of (4.10) become

$$\begin{cases} -3(k-1)(2k-3)\alpha_{k-1}^2 d_0 = 0 \\ -6(k-1)(2k-1)\alpha_{k-1}\alpha_k d_0 + \dfrac{8(k-1)(3k-1)\alpha_{k-1}^2}{k} e_0 = 0 \end{cases} \tag{4.15}$$

which show that $d_0 = e_0 = 0$. This contradicts the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$. Therefore, α_{k-1} must vanish. It has shown that $\alpha_0 = \alpha_1 = \cdots = \alpha_{k-1} = 0$. Then the $(2k-3)$-rd and the $(2k-1)$-st equations of (4.10) become

$$\begin{cases} \dfrac{1}{2}k^2(k-1)(2k-3)\alpha_k^2 b_0 = 0 \\ -3k(2k-1)\alpha_k^2 d_0 = 0 \end{cases} \tag{4.16}$$

The only possible solution is $b_0 = d_0 = 0$ with $\alpha_k \neq 0$. Therefore, we obtain

$$V_k = y^k. \tag{4.17}$$

If we assume that $\alpha_0 = 0$ for the other three cases (ii), (iii), (iv), then we only obtain the potential (4.17). Therefore, we assume that $\alpha_0 \neq 0$ for the cases (ii), (iii), (iv). Let us put $\alpha_0 = 1$.

(ii) When $\alpha_2 = 0$ and $\alpha_0 = 1$, we can see from the first equation of (4.10) that $b_0 = 0$. Then the third equation of (4.10) becomes

$$-\frac{12(2k-3)(3k-4)\alpha_3}{k-2} a_0 = 0, \tag{4.18}$$

which shows that $\alpha_3 a_0 = 0$. Then the recurrence relation (4.9) gives

$$-k(k-1)(k-2)(k-3)d_0 = 0 \quad (j = 2) \tag{4.19}$$

from which we obtain $d_0 = 0$. Let us now suppose that we have proved that $\alpha_1 = \alpha_2 = \cdots = \alpha_j = 0$ $(j \geq 2)$. Then it follows from (4.9) that α_{j+1} must vanish, up to $j = k-4$, under the condition $(a_0, e_0) \neq (0,0)$. Now, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-3} = 0$. If we assume that $\alpha_{k-2} \neq 0$ then we can see from (4.9) with $j = k-3$ that $a_0 = 0$. Then the kth equation of (4.10) reads

$$-\frac{8k(k+2)\alpha_{k-2}}{k-1} e_0 = 0, \tag{4.20}$$

which shows that $e_0 = 0$ since $\alpha_{k-2} \neq 0$. This contradicts the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$. Therefore, α_{k-2} must vanish. Next, let us assume that $\alpha_{k-1} \neq 0$. Then we

can see from (4.9) with $j = k-2$ that $a_0 = 0$. Then the $(k+1)$-st equation of (4.10) becomes

$$-4(k-1)\alpha_{k-1}e_0 = 0, \tag{4.21}$$

which shows that $e_0 = 0$ since $\alpha_{k-1} \neq 0$. This contradicts the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$. Therefore, α_{k-1} must vanish. Then it has shown that $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$. Thus, we obtain

$$V_k = x^k + \alpha_k y^k. \tag{4.22}$$

(iii) When $\alpha_2 = (k(k-1)/2)\alpha_0$ and $\alpha_0 = 1$, the third and the fourth equations of (4.10) yield

$$d_0 = -\frac{M_{32}a_0 + M_{31}b_0}{M_{33}}, \tag{4.23}$$

$$e_0 = -\frac{1}{M_{44}}\left\{\left(M_{42} + \frac{M_{43}M_{32}}{M_{33}}\right)a_0 + \left(M_{41} + \frac{M_{43}M_{31}}{M_{33}}\right)b_0\right\}. \tag{4.24}$$

Then the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$ reads $(b_0, a_0) \neq (0,0)$. We obtain

$$\alpha_j = \binom{k}{j}\left[\frac{(\sin\varphi_0)^j}{(\cos\varphi_0)^{j-2}} \pm (-1)^{k-j}\frac{(\cos\varphi_0)^j}{(\sin\varphi_0)^{j-2}}\right]; \quad \tan 2\varphi_0 = -\frac{2}{\alpha_3}\binom{k}{3} \tag{4.25}$$

from the recurrence relation (4.9) under the condition $(b_0, a_0) \neq (0,0)$. Here, we take the positive sign for even k and the negative sign for odd k. The potential with (4.25) is transformed into the separable form

$$V_k = (\sec\varphi_0)^{k-2}x^k \pm (\operatorname{cosec}\varphi_0)^{k-2}y^k \tag{4.26}$$

by the rotation of coordinates defined by

$$x \to x\cos\varphi_0 - y\sin\varphi_0, \quad y \to x\sin\varphi_0 + y\cos\varphi_0. \tag{4.27}$$

(iv) When $\alpha_2 = (3k(2k-1)/2)\alpha_0$ and $\alpha_0 = 1$, the second, the third, and the fourth equations of (4.10) yield

$$a_0 = -\frac{M_{21}}{M_{22}}b_0, \tag{4.28}$$

$$d_0 = -\frac{1}{M_{33}}\left(M_{31} - M_{32}\frac{M_{21}}{M_{22}}\right)b_0, \tag{4.29}$$

$$e_0 = -\frac{1}{M_{44}}\left\{M_{41} - M_{42}\frac{M_{21}}{M_{22}} - M_{43}\frac{1}{M_{33}}\left(M_{31} - M_{32}\frac{M_{21}}{M_{22}}\right)\right\}b_0. \tag{4.30}$$

Then the condition $(b_0, a_0, d_0, e_0) \neq (0,0,0,0)$ reads $b_0 \neq 0$. Let us put $b_0 = 1$. Then, from the recurrence relation (4.9), we can express $\alpha_4, \alpha_5, \ldots, \alpha_k$ as polynomials of α_3 whose coefficients are rational functions of k. Therefore, the rest of equations (4.10), the number of which is $2k-5$, yield algebraic equations for α_3. Now, the problem is whether the $2k-5$ algebraic equations for α_3 have common solutions or not. The two

algebraic equations of the lowest degrees, which are obtained from the fifth and the sixth equations of (4.10), are explicitly given by

$$
\begin{aligned}
&-\frac{9(k-2)k^2(k+2)(2k-1)(5k-2)G_1(k)}{4(k-1)S(k)^2}\\
&+\frac{3(k+2)(5k-2)(7k-6)G_2(k)\alpha_3^2}{4(k-2)(k-1)(2k-1)(3k-2)R(k)S(k)^2}\\
&+\frac{2(k+2)(5k-2)(7k-6)G_3(k)\alpha_3^4}{(k-2)^3(k-1)k^2(2k-1)^3(3k-2)^3R(k)S(k)^2}=0
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
&-\frac{9(k-3)k(k+2)(5k-2)G_4(k)\alpha_3}{20(k-1)(3k-2)R(k)S(k)^2}\\
&+\frac{3(k-3)(k+2)(5k-2)(7k-6)G_5(k)\alpha_3^3}{20(k-2)^2(k-1)k(2k-1)^2(3k-2)^2R(k)S(k)^2}\\
&+\frac{(k-3)(k+2)(5k-2)(7k-6)G_6(k)\alpha_3^5}{5(k-2)^4(k-1)k^3(2k-1)^4(3k-2)^4R(k)S(k)^2}=0,
\end{aligned}
\tag{4.32}
$$

where

$$
\begin{aligned}
R(k) &= 74-151k+95k^2, \quad S(k) = -124+456k-519k^2+241k^3,\\
G_1(k) &= 29952-260400k+932160k^2-1727456k^3+1706012k^4\\
&\quad -810861k^5+118529k^6+16870k^7,\\
G_2(k) &= -3170144+37193808k-190561760k^2+553816792k^3\\
&\quad -998755638k^4+1151765545k^5-843129587k^6\\
&\quad +371783811k^7-86588015k^8+7446900k^9,\\
G_3(k) &= -1361952+20130192k-133944304k^2+527295832k^3-1360307178k^4\\
&\quad +2408313485k^5-2978361002k^6+2565200757k^7-1500811081k^8\\
&\quad +563283562k^9-120201105k^{10}+10736550k^{11},\\
G_4(k) &= -54723072+687806528k-3817158400k^2+12223132176k^3\\
&\quad -24787677856k^4+32872864764k^5-28351948208k^6\\
&\quad +15180036291k^7-4446138980k^8+454376975k^9+40055750k^{10},\\
G_5(k) &= 68988096-912699392k+5346987920k^2-18143124368k^3\\
&\quad +39260441572k^4-56288554120k^5+53780522743k^6\\
&\quad -33457433369k^7+12720106145k^8-2562547775k^9\\
&\quad +186064500k^{10},\\
G_6(k) &= 56393856-913712256k+6725558432k^2-29645084416k^3\\
&\quad +86929567784k^4-178232080840k^5+261364852406k^6\\
&\quad -275416174516k^7+206176050353k^8-106294583086k^9\\
&\quad +35472298955k^{10}-6761687100k^{11}+538285500k^{12}.
\end{aligned}
$$

The quantity called the resultant determines whether given two polynomials have a common root or not. Let us consider the two polynomials

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \quad (a_0 \neq 0), \tag{4.33}$$

$$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m \quad (b_0 \neq 0). \tag{4.34}$$

The resultant of $f(x)$ and $g(x)$ is defined by the following determinant of order $(m+n)$.

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & & & & \\ & a_0 & a_1 & \cdots & a_n & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & a_0 & a_1 & \cdots & a_n & \\ b_0 & b_1 & \cdots & b_m & & & & \\ & b_0 & b_1 & \cdots & b_m & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & b_0 & b_1 & \cdots & \cdots & b_m & \end{vmatrix} \tag{4.35}$$

Here, all the blanks represent zeros. It is known that the following theorem holds.

Theorem 3 (see, for example, [21]) *The two polynomials $f(x)$ and $g(x)$ have a common root if and only if $R(f,g) = 0$.*

See Appendix B for the proof. The resultant of the two algebraic equations (4.31) and (4.32) is computed to be [22]

$$\begin{aligned}
&-\frac{289340}{4096} \\
&\times \frac{(k+2)^9(k-3)^4(k-4)^2(2k-3)^4(3k-4)^4(3k-1)^4(5k-6)^2(5k-2)^{29}(7k-6)^4}{k^2(k-1)^9(k-2)^{11}(2k-1)^{11}(3k-2)^{16}R(k)^6 S(k)^{10}} \\
&\times (29952 - 260400k + 932160k^2 - 1727456k^3 \\
&\qquad + 1706012k^4 - 810861k^5 + 118529k^6 + 16870k^7) \\
&\times (-2787216 + 23021920k - 82567904k^2 + 167598204k^3 \\
&\qquad - 211881739k^4 + 173025983k^5 - 91304549k^6 \\
&\qquad + 30037885k^7 - 5594600k^8 + 450000k^9)^2,
\end{aligned} \tag{4.36}$$

which does not vanish for $k \geq 5$. Then it is concluded from theorem 3 that the two algebraic equations (4.31) and (4.32) do not have any common solutions. Therefore, there exist no common solutions among the $2k-5$ algebraic equations. This means that we have no solutions for $\alpha_2 = (3k(2k-1)/2)\alpha_0$ $(\alpha_0 = 1)$.

From the above arguments, we see that the case 5 considered here only yields the separable potential of the form

$$V_k = Ax^k + By^k, \tag{4.37}$$

i.e. there exist no new integrable potentials.

5. A list of integrable homogeneous polynomial potentials

Although the computations in the previous section were performed under the implicit assumption that the degree of the potential, k, is greater than 4, we can carry out the computations for the cases $k = 3$ and $k = 4$ in the same manner. For $k = 3$, the expressions of M_{ij} are given by

$$M_{11} = \frac{(3\alpha_0 - \alpha_2)(45\alpha_0 - 2\alpha_2)}{3},$$
$$M_{21} = -30\alpha_3(3\alpha_0 - \alpha_2), \quad M_{22} = -\frac{112\alpha_2(3\alpha_0 - \alpha_2)}{3},$$
$$M_{31} = -9\alpha_0\alpha_2 - 7\alpha_2^2 + 45\alpha_3^2, \quad M_{32} = 60\alpha_2\alpha_3, \quad M_{33} = -9(\alpha_0 - 2\alpha_2)(5\alpha_0 - \alpha_2),$$
$$M_{41} = -30\alpha_2\alpha_3, \quad M_{42} = -40\alpha_2^2, \quad M_{43} = 30\alpha_3(3\alpha_0 - 2\alpha_2),$$
$$M_{44} = -\frac{8\alpha_2(3\alpha_0 - 16\alpha_2)}{3}.$$

Then the condition that the product $M_{11}M_{22}M_{33}M_{44}$ vanishes gives the following relations between α_0 and α_2.

$$\alpha_2 = 0, \quad \alpha_2 = 3\alpha_0, \quad \alpha_2 = \frac{45}{2}\alpha_0, \quad \alpha_2 = \frac{1}{2}\alpha_0, \quad \alpha_2 = 5\alpha_0, \quad \alpha_2 = \frac{3}{16}\alpha_0. \tag{5.1}$$

When $\alpha_2 = 0$, we obtain the separable potential

$$V_3 = \alpha_0 x^3 + \alpha_3 y^3 \tag{5.2}$$

and when $\alpha_2 = 3\alpha_0$ ($\alpha_0 = 1$), we obtain the potential

$$V_3 = x^3 + 3xy^2 + \alpha_3 y^3, \tag{5.3}$$

which is transformed into the form of (5.2) by the rotation of coordinates defined by (4.27). When $\alpha_2 = (45/2)\alpha_0$ ($\alpha_0 = 1$), we obtain an algebraic equation for α_3, which is given by (4.31) with $k = 3$, after the same computations as in the previous section. Then we obtain the potentials

$$V_3 = x^3 + \frac{45}{2}xy^2 + \frac{17\sqrt{14}\,i}{2}y^3, \tag{5.4}$$
$$V_3 = x^3 + \frac{45}{2}xy^2 - \frac{27\sqrt{3}\,i}{2}y^3. \tag{5.5}$$

When $\alpha_2 = (1/2)\alpha_0$ ($\alpha_0 = 1$), we obtain the potential

$$V_3 = x^3 + \frac{1}{2}xy^2 + \frac{\sqrt{3}\,i}{18}y^3. \tag{5.6}$$

When $\alpha_2 = 5\alpha_0$ ($\alpha_0 = 1$), we obtain the potential

$$V_3 = x^3 + 5xy^2 + \frac{22\sqrt{3}\,i}{9}y^3. \tag{5.7}$$

The potentials (5.5), (5.6), (5.7) are transformed into each other by proper rotations of coordinates. When $\alpha_2 = (3/16)\alpha_0$ ($\alpha_0 = 1$), we obtain the potential

$$V_3 = x^3 + \frac{3}{16}xy^2. \tag{5.8}$$

The potentials (5.4), (5.8) are transformed into each other by proper rotations of coordinates.

For $k = 4$, the expressions of M_{ij} are given by

$$M_{11} = \frac{(6\alpha_0 - \alpha_2)(42\alpha_0 - \alpha_2)}{3},$$
$$M_{21} = -11\alpha_3(6\alpha_0 - \alpha_2), \quad M_{22} = -\frac{80\alpha_2(6\alpha_0 - \alpha_2)}{3},$$
$$M_{31} = \frac{540\alpha_0\alpha_2 - 16\alpha_2^2 + 99\alpha_3^2 - 1512\alpha_0\alpha_4 + 228\alpha_2\alpha_4}{6},$$
$$M_{32} = -80\alpha_3(3\alpha_0 - \alpha_2), \quad M_{33} = 80\alpha_0\alpha_2,$$
$$M_{41} = -2\alpha_3(6\alpha_0 + 37\alpha_2 - 48\alpha_4), \quad M_{42} = -\frac{8(32\alpha_2^2 - 27\alpha_3^2 - 24\alpha_2\alpha_4)}{3},$$
$$M_{43} = 24\alpha_3(7\alpha_0 - \alpha_2), \quad M_{44} = -\frac{64\alpha_2(3\alpha_0 - 4\alpha_2)}{3}.$$

Then the condition that the product $M_{11}M_{22}M_{33}M_{44}$ vanishes gives the following relations between α_0 and α_2.

$$\alpha_0 = 0, \quad \alpha_2 = 0, \quad \alpha_2 = 6\alpha_0, \quad \alpha_2 = 42\alpha_0, \quad \alpha_2 = \frac{3}{4}\alpha_0. \tag{5.9}$$

When α_0, we obtain the potential

$$V_4 = y^4, \tag{5.10}$$

and when $\alpha_2 = 0$ ($\alpha_0 = 1$), we obtain the potential

$$V_4 = x^4 + \alpha_4 y^4. \tag{5.11}$$

When $\alpha_2 = 6\alpha_0$ ($\alpha_0 = 1$), we obtain the potentials

$$V_4 = x^4 + 6x^2y^2 + \alpha_3 xy^3 + \frac{16 + \alpha_3^2}{16}y^4, \tag{5.12}$$
$$V_4 = x^4 + 6x^2y^2 + 8y^4. \tag{5.13}$$

The potential (5.12) is transformed into the form of (5.11) by the rotation of coordinates (4.27). When $\alpha_2 = 42\alpha_0$ ($\alpha_0 = 1$), we obtain three algebraic equations for α_3, two of which are given by (4.31) and (4.32) with $k = 4$, after the same computations as in the previous section. They have common solutions, which is indicated by the fact that the resultant (4.36) vanishes when $k = 4$. Then we obtain the potential

$$V_4 = x^4 + 42x^2y^2 + 28\sqrt{10}\, ixy^3 - 48y^4. \tag{5.14}$$

When $\alpha_2 = (3/4)\alpha_0$ ($\alpha_0 = 1$), we obtain the potential

$$V_4 = x^4 + \frac{3}{4}x^2y^2 + \frac{1}{8}y^4. \tag{5.15}$$

The potentials (5.13), (5.14), (5.15) are transformed into one another by proper rotations of coordinates.

As a consequence, we obtain the following list of integrable 2D homogeneous polynomial potentials with a polynomial first integral of order at most 4 in the momenta.

- with a polynomial first integral linear in the momenta

$$V_k = (x^2 + y^2)^{k/2}, \quad k = \text{even}. \tag{5.16}$$

- with a polynomial first integral quadratic in the momenta

$$V_k = \frac{1}{r}\left[\left(\frac{r+x}{2}\right)^{k+1} + (-1)^k\left(\frac{r-x}{2}\right)^{k+1}\right], \quad V_k = Ax^k + By^k. \tag{5.17}$$

- with a polynomial first integral quartic in the momenta

$$V_3 = x^3 + \frac{3}{16}xy^2, \quad V_3 = x^3 + \frac{1}{2}xy^2 + \frac{\sqrt{3}\,i}{18}y^3, \quad V_4 = x^4 + \frac{3}{4}x^2y^2 + \frac{1}{8}y^4. \tag{5.18}$$

As far as the authors know, no one has discovered any polynomial first integral which is *genuinely* quintic or higher orders in the momenta. It is still an open problem whether or not there exist such polynomial first integrals.

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Appendix A. Proof of the properties 1 and 2

Appendix A.1. Time reflection symmetry – proof of the property 1

The system (1.1) is invariant by the time reflection

$$t \to -t, \quad x \to x, \quad y \to y, \quad p_x \to -p_x, \quad p_y \to -p_y. \tag{A.1}$$

If $\Phi(x, y, p_x, p_y)$ is a first integral of the system (1.1), then $\Phi(x, y, -p_x, -p_y)$ is also a first integral because of the time reflection symmetry of the system. Note here that every first integral Φ can be decomposed into the sum of Φ_{even} and Φ_{odd}, given by

$$\Phi_{\text{even}} = \frac{\Phi(x, y, p_x, p_y) + \Phi(x, y, -p_x, -p_y)}{2} \tag{A.2}$$

and

$$\Phi_{\text{odd}} = \frac{\Phi(x, y, p_x, p_y) - \Phi(x, y, -p_x, -p_y)}{2}. \tag{A.3}$$

We can see that Φ_{even} is a first integral which is even in the momenta and that Φ_{odd} is a first integral which is odd in the momenta. That is, if $\Phi = \Phi_{\text{even}} + \Phi_{\text{odd}}$ is a first integral of the system (1.1), then Φ_{even} and Φ_{odd} are also first integrals. Therefore, we can assume that a first integral is either even or odd in the momenta from the beginning.

Appendix A.2. Scale invariance – proof of the property 2

The system (1.1) with a homogeneous polynomial potential of degree k is invariant by the scale transformation

$$t \to \sigma^{-1} t, \quad x \to \sigma^{2/(k-2)} x, \quad y \to \sigma^{2/(k-2)} y, \quad p_x \to \sigma^{k/(k-2)} p_x, \quad p_y \to \sigma^{k/(k-2)} p_y. \tag{A.4}$$

In general, a system of differential equations

$$\frac{dx_i}{dt} = F_i(x_1, x_2, \ldots, x_n), \quad (i = 1, 2, \ldots, N) \tag{A.5}$$

is called a scale invariant system if it is invariant by the scale transformation

$$t \to \sigma^{-1} t, \quad x_i \to \sigma^{g_i} x_i, \quad (i = 1, 2, \ldots, N) \tag{A.6}$$

with an arbitrary parameter σ and proper constants g_i. A function $\Phi(x_1, x_2, \ldots, x_N)$ is said to be weighted-homogeneous with a weight M if it satisfies

$$\Phi(\sigma^{g_1} x_1, \sigma^{g_2} x_2, \ldots, \sigma^{g_N} x_N) = \sigma^M \Phi(x_1, x_2, \ldots, x_N). \tag{A.7}$$

Suppose now that the scale invariant system (A.5) has a polynomial first integral Φ, which can be written in the form

$$\Phi = \sum_m \Phi_m, \tag{A.8}$$

where each polynomial Φ_m is weighted-homogeneous with a weight m. The scale transformation (A.6) transforms the first integral (A.8) into

$$\Phi' = \sum_m \sigma^m \Phi_m, \tag{A.9}$$

which is again a first integral for an arbitrary σ because of the scale invariance of the system. Then, it is concluded that each polynomial Φ_m is a first integral. Therefore, we can assume that a polynomial first integral is weighted-homogeneous from the beginning.

Appendix B. Resultant – proof of theorem 3

The resultant is an algebraic tool for elimination of a variable between two algebraic equations and gives the condition that the two algebraic equations have a common root (theorem 3). See [21] for more details of the resultant and its applications.

Proof of theorem 3: Suppose that the two polynomials have a common root, say α. Then the following simultaneous algebraic equations hold.

$$\left\{\begin{array}{rl}
\alpha^{m-1} f(\alpha) & = a_0 \alpha^{m+n-1} + a_1 \alpha^{m+n-2} + \cdots + a_n \alpha^{m-1} = 0 \\
\alpha^{m-2} f(\alpha) & = a_0 \alpha^{m+n-2} + a_1 \alpha^{m+n-3} + \cdots + a_n \alpha^{m-2} = 0 \\
 & \vdots \\
f(\alpha) & = a_0 \alpha^{n} + a_1 \alpha^{n-1} + \cdots + a_n = 0 \\
\alpha^{n-1} g(\alpha) & = b_0 \alpha^{m+n-1} + b_1 \alpha^{m+n-2} + \cdots + b_m \alpha^{n-1} = 0 \\
\alpha^{n-2} g(\alpha) & = b_0 \alpha^{m+n-2} + b_1 \alpha^{m+n-3} + \cdots + b_m \alpha^{n-2} = 0 \\
 & \vdots \\
g(\alpha) & = b_0 \alpha^{m} + b_1 \alpha^{m-1} + \cdots + b_m = 0
\end{array}\right. \tag{B.1}$$

If we multiply the lth column of the resultant by α^l and add them to the $(m+n)$-th column, then all the elements of the $(m+n)$-th column of the resultant vanish because of (B.1). Therefore, $R(f,g)=0$.

Let us next assume that $R(f,g)=0$. Then the $m+n$ row vectors of the resultant are linear dependent. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ be the row vectors of the resultant. Then the relation

$$\sum_{i=1}^{m} c_i \boldsymbol{a}_i + \sum_{j=1}^{n} d_j \boldsymbol{b}_j = 0 \tag{B.2}$$

holds with $(c_1, \ldots, d_n) \neq (0, \ldots, 0)$. Multiplying the lth element of (B.2) by x^{m+n-l} and adding them up, we have

$$\sum_{i=1}^{m} c_i x^{m-i} f(x) + \sum_{j=1}^{n} d_j x^{n-j} g(x) = 0, \tag{B.3}$$

which is reduced to

$$h(x) f(x) = -k(x) g(x), \tag{B.4}$$

where $\sum_{i=1}^{m} c_i x^{m-i} = h(x), \sum_{j=1}^{n} d_j x^{n-j} = k(x)$. Let $\deg f(x)$ denote the degree of a polynomial $f(x)$. Then, we have the relation

$$\deg k(x) \leq n-1 < n = \deg f(x). \tag{B.5}$$

If $f(x)$ and $g(x)$ have no common factors, then $f(x)$ must be a factor of $k(x)$ because of (B.4). This contradicts equation (B.5). Therefore, $f(x)$ and $g(x)$ have at least one common factor, i.e. they have at least one common root. This completes the proof of theorem 3.

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